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PICARD-FUCHS EQUATIONS AND MIRROR MAPS FOR HYPERSURFACES

DAVID R. MORRISON

Abstract. We describe a strategy for computing Yukawa couplings and the mirror map, based on the Picard-Fuchs equation. (Our strategy is a variant of the method used by Candelas, de la Ossa, Green, and Parkes [5] in the case of quintic hypersurfaces.) We then explain a technique of Griffiths [14] which can be used to compute the Picard-Fuchs equations of hypersurfaces. Finally, we carry out the computation for four specific examples (including quintic hypersurfaces, previously done by Candelas et al. [5]). This yields predictions for the number of rational curves of various degrees on certain hypersurfaces in weighted projective spaces. Some of these predictions have been confirmed by classical techniques in algebraic geometry.

Introduction

The phenomenon of mirror symmetry dramatically caught the attention of mathematicians with the recent work of P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes [5]. Starting with a particular pair of "mirror manifolds", calculating certain period integrals, interpreting the results as Yukawa couplings, and then re-interpreting those results in light of the "mirror manifold" phenomenon, Candelas et al. were able to give predictions for the numbers of rational curves of various degrees on the general quintic threefold. In fact, algebraic geometers have had a difficult time verifying these predictions, but all successful attempts to calculate the numbers of curves have eventually confirmed the predictions.

What is so striking about this work is that the calculation which predicts the numbers of rational curves on quintic threefolds is in reality a calculation about the variation of Hodge structure on a completely *different* family of Calabi-Yau threefolds. An asymptotic expansion is made of a function which comes from that variation, and the coefficients in the expansion are then used to predict numbers of rational curves.

In [21], we interpreted the calculation of Candelas et al. [5] in terms of variation of Hodge structure. Here we take a more down to earth

approach, and work directly with period integrals and their properties. (This is perhaps closer in spirit to the original paper.) We have found a way to modify the computational strategy employed in [5]. Our modified method computes a bit less (there are two unknown "constants of integration"), but it is easier to actually carry out the computation. We in fact carry it out in three new examples. This leads to new predictions about numbers of rational curves on certain Calabi-Yau threefolds.

Our strategy for computing Yukawa couplings is based on the Picard-Fuchs equation for the periods of a one-parameter family of algebraic varieties. We explain in sections 1 and 2 how this equation can be used to compute Yukawa couplings and the mirror map for a family of Calabi-Yau threefolds with $h^{2,1} = 1$. We then go on in section 3 to review a method of Griffiths [14] for calculating Picard-Fuchs equations of hypersurfaces. Related ideas have also been introduced into the physics literature in [2, 4, 12, 19].

In sections 4 and 5, we carry out the computation in four examples, including the quintic hypersurface. The resulting predictions about numbers of rational curves are discussed in section 6.

1. The Picard-Fuchs equation and monodromy

Let $\bar{\pi} : \overline{\mathcal{X}} \to \overline{C}$ be a family of n-dimensional projective algebraic varieties, parameterized by a compact Riemann surface $\overline{C}$. Let $C \subset \overline{C}$ be an open subset such that the induced family $\pi : \mathcal{X} \to C$ has smooth fibers. If we choose topological n-cycles $\gamma_0, \ldots, \gamma_{r-1}$ which give a basis for the n^{th} homology of one particular fiber X_0, and choose a holomorphic n-form ω on X_0, then the *periods* of ω are the integrals

$$\int_{\gamma_0} \omega, \ldots, \int_{\gamma_{r-1}} \omega.$$

Since the fibration $\pi : \mathcal{X} \to C$ is differentiably locally trivial, a local trivialization can be used to extend the cycles γ_i from X_0 to cycles $\gamma_i(z)$ on X_z which depend on z, where z is a local coordinate on C. The holomorphic n-form ω can also be extended to a family of n-forms $\omega(z)$ which depend on the parameter z. If this is done in an algebraic way, then $\omega(z)$ extends to a meromorphic family of n-forms (i.e. poles are allowed) over the entire space $\overline{\mathcal{X}}$.

The cycles $\gamma_i(z)$ determine homology classes which are locally constant in z. However, an attempt to extend these cycles globally will typically lead to monodromy: for each closed path in C, there will be some linear map T represented by a matrix T_{ij} such that transporting γ_i along the path produces at the end a cycle homologous to $\sum T_{ij}\gamma_j$.

The same phenomenon will hold for the periods: for a globally defined meromorphic family of n-forms $\omega(z)$, the local periods $\int_{\gamma_i(z)} \omega(z)$ extend by analytic continuation to multiple-valued functions of z, transforming according to the same monodromy transformations T as do the homology classes of the cycles.

The periods $\int_{\gamma(z)} \omega(z)$ satisfy an ordinary differential equation called the *Picard-Fuchs equation* of ω. The existence of this equation can be explained as follows. Choose a local coordinate z on some open set $U \subset C$, and consider the vector

$$v_j(z) := [\frac{d^j}{dz^j} \int_{\gamma_0(z)} \omega(z), \dots, \frac{d^j}{dz^j} \int_{\gamma_{r-1}(z)} \omega(z)] \in \mathbb{C}^r.$$

For generic values of the parameter z, the dimensions

$$d_j(z) := \dim(\mathrm{span}\{v_0(z), \dots, v_j(z)\})$$

must be constant. Since $d_j(z) \leq r$, these spaces cannot continue to grow indefinitely. There will thus be a smallest s such that

$$v_s(z) \in \mathrm{span}\{v_0(z), \dots, v_{s-1}(z)\}$$

(for generic z). We can write

$$v_s(z) = -\sum_{j=0}^{s-1} C_j(z) v_j(z)$$

with the coefficients $C_j(z)$ depending on z. The *Picard-Fuchs equation*, satisfied by all the periods of $\omega(z)$, is then

$$\frac{d^s f}{dz^s} + \sum_{j=0}^{s-1} C_j(z) \frac{d^j f}{dz^j} = 0. \tag{1}$$

The precise form of the equation depends on both the local coordinate z on C, and the choice of holomorphic form $\omega(z)$. Note that the coefficients $C_j(z)$ may acquire singularities at special values of z.

When we approach a point P in $\overline{C} - C$, the Picard-Fuchs equation has (at worst) a *regular singular point* at P [15, 17, 8]. If we choose a parameter z which is centered at P (that is, $z = 0$ at P), then the coefficients $C_j(z)$ in the Picard-Fuchs equation typically will have poles at $z = 0$. However, if we multiply the Picard-Fuchs operator

$$\frac{d^s}{dz^s} + \sum_{j=0}^{s-1} C_j(z) \frac{d^j}{dz^j} \tag{2}$$

by z^s and rewrite the result in the form

$$(z\frac{d}{dz})^s + \sum_{j=0}^{s-1} B_j(z) (z\frac{d}{dz})^j \tag{3}$$

then the new coefficients $B_j(z)$ are holomorphic functions of z. (This is one of several equivalent definitions of "regular singular point".) We call eq. (3) the *logarithmic form* of the Picard-Fuchs operator.

The structure of ordinary differential equations with regular singular points is a classical topic in differential equations: a convenient reference is [7]. We can rewrite eq. (1) as a system of first-order equations, using the logarithmic form eq. (3), as follows: let

$$A(z) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -B_0(z) & -B_1(z) & \dots & \dots & -B_{s-1}(z) \end{bmatrix}. \tag{4}$$

Then solutions $f(z)$ to the equation eq. (1) are equivalent to solution vectors

$$w(z) = \begin{bmatrix} f(z) \\ z\frac{d}{dz}f(z) \\ \vdots \\ (z\frac{d}{dz})^{s-1}f(z) \end{bmatrix}$$

of the matrix equation

$$z\frac{d}{dz}w(z) = A(z)w(z). \tag{5}$$

For a matrix equation such as eq. (5), the facts are these (see [7]). There is a constant $s \times s$ matrix R and a $s \times s$ matrix $S(z)$ of (single-valued) functions of z, regular near $z = 0$, such that

$$\Phi(z) = S(z) \cdot z^R$$

is a *fundamental matrix* for the system. This means that the columns of $\Phi(z)$ are a basis for the space of solutions at each nonsingular point $z \neq 0$. The multiple-valuedness of the solutions has all been put into R, since

$$z^R := e^{(log z)R} = I + (\log z)R + \frac{(\log z)^2}{2!}R^2 + \cdots$$

is a multiple-valued matrix function of z. The local monodromy on the solutions given by analytic continuation along a path winding once around $z = 0$ in a counterclockwise direction is given by $e^{2\pi i R}$ (with respect to the basis given by the columns of Φ). The matrix R is by no means unique.

Theorem . *Suppose that* $z\frac{d}{dz}w(z) = A(z)w(z)$ *is a system of ordinary differential equations with a regular singular point at* $z = 0$. *Suppose*

that distinct eigenvalues of $A(0)$ do not differ by integers. Then there is a fundamental matrix of the form

$$\Phi(z) = S(z) \cdot z^{A(0)}$$

and $S(z)$ can be obtained as a power series

$$S(z) = S_0 + S_1 z + S_2 z^2 + \cdots$$

by recursively solving the equation

$$z\frac{d}{dz}S(z) + S(z) \cdot A(0) = A(z) \cdot S(z)$$

for the coefficient matrices S_j. Moreover, any such series solution converges in a neighborhood of $z = 0$.

A proof can be found in [7], together with methods for treating the case in which eigenvalues of $A(0)$ *do* differ by integers.

We will be particularly interested in systems with *unipotent monodromy*: by definition, this means that $e^{2\pi i R}$ is a unipotent matrix, so that $(e^{2\pi i R} - I)^m \neq 0$, $(e^{2\pi i R} - I)^{m+1} = 0$ for some m called the *index*.

Corollary . *Suppose that $(z\frac{d}{dz})^s f(z) + \sum_{j=0}^{s-1} B_j(z)(z\frac{d}{dz})^j f(z)$ is an ordinary differential equation with a regular singular point at $z = 0$. If $B_j(0) = 0$ for all j, then the solutions of this equation have unipotent monodromy of index s.*

The corollary follows by calculating with eq. (4), setting $z = 0$ and $B_j(0) = 0$ to produce

$$e^{2\pi i A(0)} = \begin{bmatrix} 1 & 2\pi i & \frac{(2\pi i)^2}{2!} & \dots & \frac{(2\pi i)^{s-1}}{(s-1)!} \\ & 1 & 2\pi i & \dots & \frac{(2\pi i)^{s-2}}{(s-2)!} \\ & & \ddots & & \vdots \\ & & & 1 & 2\pi i \\ & & & & 1 \end{bmatrix}.$$

2. Computing the mirror map

Recall that a *Calabi-Yau manifold* is a compact Kähler manifold X of complex dimension n which has trivial canonical bundle, such that the Hodge numbers $h^{k,0}$ vanish for $0 < k < n$. Thanks to a celebrated theorem of Yau [27], every such manifold admits Ricci-flat Kähler metrics.

Suppose now that $\pi : \mathcal{X} \to C$ is a family of Calabi-Yau threefolds with $h^{2,1}(X) = 1$, which is not a locally constant family. The third cohomology group $H^3(X)$ has dimension $r = 4$. It follows that the

Picard-Fuchs equation has order at most 4. (In fact, it is not difficult to show that it has order exactly 4.)

Let z be a coordinate on $\overline{C}$ centered at a point $P \in \overline{C} - C$. We say that P *is a point at which the monodromy is maximally unipotent* if the monodromy is unipotent of index 4. As we have seen in the corollary, if $B_j(0) = 0$ in the logarithmic form of the Picard-Fuchs equation, $z = 0$ will be such a point. We will assume for simplicity that our points of maximally unipotent monodromy have this form, leaving appropriate modifications for the general case to the reader.

We review the calculation of the Yukawa coupling, following [5]. Let $\omega(z)$ be a family of n-forms, and let

$$W_k := \int_{X_z} \omega(z) \wedge \frac{d^k}{dz^k}\omega(z).$$

A fundamental principle from the theory of variation of Hodge structure (cf. [16]) implies that W_0, W_1, and W_2 all vanish. The *Yukawa coupling* is the first non-vanishing term W_3. Candelas et al. show that the Yukawa coupling W_3 satisfies the differential equation

$$\frac{dW_3(z)}{dz} = -\frac{1}{2}C_3(z)W_3(z),$$

where $C_3(z)$ is a coefficient in the Picard-Fuchs equation (1).

The Yukawa coupling as defined clearly depends on the "gauge", that is, on the choice of holomorphic 3-form $\omega(z)$. If fact, if we alter the gauge by $\omega(z) \mapsto f(z)\omega(z)$, then W_k transforms as

$$W_k \mapsto f(z)\sum_{j=0}^{k}\binom{k}{j}\frac{d^j f(z)}{dz^j}W_{k-j}.$$

Since $W_0 = W_1 = W_2 = 0$, the change in the Yukawa coupling W_3 is simply $W_3 \mapsto f(z)^2 W_3$.

The Yukawa coupling also depends on the choice of coordinate z, and in fact is often denoted by κ_{zzz}. If we change coordinates from z to w, we must change the differentiation operator from d/dz to d/dw. The chain rule then imples that

$$\kappa_{www} = \left(\frac{dz}{dw}\right)^3 \kappa_{zzz}.$$

Candelas et al. [5] use physical arguments to set the gauge in this calculation, and to find an appropriate (multiple-valued) parameter t with which to compute. (The associated differentiation operator d/dt is single-valued.) What will be important for us are the following observations about their results.

The gauge used by Candelas et al. determines a family of meromorphic n-forms $\widetilde{\omega}(z)$ with the property that the period function

$$\int_\gamma \widetilde{\omega}(z) \equiv 1$$

for some cycle γ. Moreover, the parameter t determined by Candelas et al. is a parameter defined in an angular sector near $z = 0$ which has two crucial properties:

1. If we analytically continue along a simple loop around $z = 0$ in the counterclockwise direction, t becomes $t + 1$. (It will be convenient to also introduce $q = e^{2\pi i t}$, which remains single-valued near $z = 0$.)
2. There are cycles γ_0 and γ_1 such that $\int_{\gamma_0} \omega(z)$ is single valued near $z = 0$, and
$$t = \frac{\int_{\gamma_1} \omega(z)}{\int_{\gamma_0} \omega(z)}$$
in an angular sector near $z = 0$.

Each period function $\int_\gamma \omega(z)$ is a solution to the Picard-Fuchs equation of the family. Translating the results of the previous section into the present context, we obtain the following:

Lemma . *Suppose that $z = 0$ is a point of maximally unipotent monodromy such that $B_j(0) = 0$, where $B_j(z)$ are the coefficients in the logarithmic form of the Picard-Fuchs equation. Then*

1. *There is a period function for $\omega(z)$,*
$$f_0(z) := \int_{\gamma_0} \omega(z)$$
which is single-valued near $z = 0$. This period function is unique up to multiplication by a constant. (This implies that the cycle γ_0 is also unique up to a constant multiple.)

   In particular, the family of meromorphic n-forms
$$\widetilde{\omega}(z) := \frac{\omega(z)}{\int_{\gamma_0} \omega(z)}$$
will have the property that
$$\int_\gamma \widetilde{\omega}(z) \equiv 1$$
for some γ, and it is the unique such family up to constant multiple.

2. *Fixing a choice of period function $f_0(z)$ as in part (1), there is a period function*
$$f_1(z) := \int_{\gamma_1} \omega(z)$$
such that $\varphi(z) := f_1(z)/f_0(z)$ transforms as
$$\varphi(z) \mapsto \varphi(z) + 1$$
upon transport around $z = 0$ in the counterclockwise direction. The ratio $\varphi(z)$ is unique up to the addition of a constant.

This, then, is our alternate strategy for computing the Yukawa coupling: we find solutions of the Picard-Fuchs equation which have the properties specified in the lemma, and we use those to fix the gauge and specify the natural parameter, up to two unknown constants of integration.

3. Picard-Fuchs equations for hypersurfaces

We now review a method of Griffiths [14] for describing the cohomology of a hypersurface, which can be used to determine the Picard-Fuchs equation of a one-parameter family of hypersurfaces. Calculations of this sort were earlier made by Dwork [10, Sec. 8]. Griffiths' method was extended to the weighted projective case by Steenbrink [26] and Dolgachev [9], who we follow.

We denote a weighted projective n-space by $\mathbb{P}^{(k_0,\dots,k_n)}$, where $k_0, \dots, k_n$ are the weights of the variables $x_0, \dots, x_n$. Weighted homogeneous polynomials can be identified with the aid of the Euler vector field
$$\theta = \sum k_j x_j \frac{\partial}{\partial x_j}$$
which has the property that $\theta P = (\deg P) \cdot P$ for any weighted homogeneous polynomial P. Contracting the volume from on $\mathbb{C}^{n+1}$ with θ produces the fundamental weighted homogeneous differential form (of "weight" $k := \sum k_j$)
$$\Omega := \sum_{j=0}^{n} (-1)^j k_j x_j \, dx_0 \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n.$$
Rational differentials of degree n on $\mathbb{P}^{(k_0,\dots,k_n)}$ can be described as expressions $P\Omega/Q$, where P and Q are weighted homogeneous polynomials with $\deg P + k = \deg Q$.

Suppose that Q is a weighted homogeneous polynomial defining a quasismooth hypersurface $\mathcal{Q} \subset \mathbb{P}^{(k_0,\dots,k_n)}$. (That is, $Q = 0$ defines a hypersurface in $\mathbb{C}^{n+1}$ which is smooth away from the origin.) The middle cohomology of $\mathcal{Q}$ is then described by means of differential forms

with poles (of all orders) along $\mathcal{Q}$. Each such form $P\Omega/Q^\ell$ is made into a cohomology class by a "residue" construction: for an $(n-1)$-cycle γ on $\mathcal{Q}$, the tube over γ (an S^1-bundle inside the (complex) normal bundle of $\mathcal{Q}$) is an n-cycle Γ on $\mathbb{P}^{(k_0,\dots,k_n)}$ disjoint from $\mathcal{Q}$. We can then define the residue of $P\Omega/Q^\ell$ by

$$\int_\gamma \operatorname{Res}_{\mathcal{Q}}\left(\frac{P\Omega}{Q^\ell}\right) = \frac{1}{2\pi i}\int_\Gamma \frac{P\Omega}{Q^\ell}.$$

Since altering $P\Omega/Q^\ell$ by an exact differential does not change the value of these integrals, we see that the cohomology of $\mathcal{Q}$ is represented by equivalence classes of rational differential forms $P\Omega/Q^\ell$ modulo exact forms.

Here is Griffiths' "reduction of pole order" calculation which shows how to reduce modulo exact forms in practice. Let Q and A_j be weighted homogeneous polynomials, with $\deg Q = d$, $\deg A_j = \ell d + k_j - k$. Define

$$\varphi = \frac{1}{Q^\ell}\sum_{i<j}(k_i x_i A_j - k_j x_j A_i)\,dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge \widehat{dx_j} \wedge \cdots \wedge dx_n$$

and then calculate

$$d\varphi = \frac{\left(\ell \sum A_j \frac{\partial Q}{\partial x_j} - Q \sum \frac{\partial A_j}{\partial x_j}\right)\Omega}{Q^{\ell+1}} = \frac{\ell \sum A_j \frac{\partial Q}{\partial x_j}\Omega}{Q^{\ell+1}} - \frac{\sum \frac{\partial A_j}{\partial x_j}\Omega}{Q^\ell}. \tag{6}$$

Thus, any form whose numerator lies in the Jacobian ideal $J = (\partial Q/\partial x_0, \ldots, \partial Q/\partial X_n)$ is equivalent (modulo exact forms) to a form with smaller pole order.

This idea can be used to calculate Picard-Fuchs equations as follows. The cycles Γ do not change (in homology) when z varies locally. So we can differentiate under the integral sign

$$\frac{d^k}{dz^k}\int_\gamma \operatorname{Res}_{\mathcal{Q}}\left(\frac{P\Omega}{Q^\ell}\right) = \frac{1}{2\pi i}\int_\Gamma \frac{d^k}{dz^k}\left(\frac{P\Omega}{Q^\ell}\right)$$

when Q depends on a parameter z. (Note that Ω is independent of z.) The Picard-Fuchs operator (2) will have the property that

$$\left(\frac{d^s}{dz^s} + \sum_{j=0}^{s-1} C_j(z)\frac{d^j}{dz^j}\right)\left(\frac{P\Omega}{Q}\right) = d\varphi$$

is an exact form. To find it, take successive z-derivatives of the integrand $P\Omega/Q$ and use the reduction of order of pole formula [14] to determine a linear relation among those derivatives, modulo exact forms.

4. Examples: Picard-Fuchs equations

We will calculate the Picard-Fuchs equations for certain one-parameter families of Calabi-Yau threefolds. Our choice of families is motivated by the mirror construction of Greene and Plesser [13].

We choose weights $k_0, \dots, k_4$ with $k_0 \ge k_1 \ge \cdots \ge k_4$ for a weighted projective 4-space such that $d_j := k/k_j$ is an integer, where $k := \sum k_j$. We also assume that $\gcd\{k_j \mid j \ne j_0\} = 1$ for every j_0. These assumptions then imply that $k = \operatorname{lcm}\{d_j\}$.

Consider the pencil of hypersurfaces $\mathcal{Q}_\psi \subset \mathbb{P}^{(k_0,\dots,k_4)}$ defined by $Q(x,\psi) = 0$, where

$$Q(x,\psi) := \sum_{j=0}^{4} x_j^{d_j} - k\psi \prod_{j=0}^{4} x_j.$$

This pencil has a natural group of diagonal automorphisms preserving the holomorphic 3-form. To define it, let $\boldsymbol{\mu_m}$ denote the multiplicative group of m^{th} roots of unity (considered as a subgroup of $\mathbb{C}^\times$), and let

$$G = (\boldsymbol{\mu_{d_0}} \times \cdots \times \boldsymbol{\mu_{d_4}})/\boldsymbol{\mu_k},$$

where we embed $\boldsymbol{\mu_k}$ in $\boldsymbol{\mu_{d_0}} \times \cdots \times \boldsymbol{\mu_{d_4}}$ by

$$\alpha \mapsto (\alpha^{k_0}, \dots, \alpha^{k_4}).$$

Note that since $\sum k_j = k$, the formula

$$f(\alpha_0, \dots, \alpha_4) = (\textstyle\prod \alpha_j)^{-1}$$

determines a well-defined homomorphism $f : G \to \mathbb{C}^\times$. Let $G_0 = \ker(f)$.

We can regard $Q(x,\psi) = 0$ as defining a hypersurface $\mathcal{Q} \subset \mathbb{P}^{(k_0,\dots,k_4)} \times \mathbb{C}$. The group G acts on $\mathbb{P}^{(k_0,\dots,k_4)} \times \mathbb{C}$ by

$$(x_0, \dots, x_4; \psi) \mapsto (\alpha_0 x_0, \dots, \alpha_4 x_4; f(\alpha)\psi)$$

for $\alpha = (\alpha_0, \dots, \alpha_4) \in G$. The polynomial $Q(x,\psi)$ is invariant under this action. Thus, the action preserves $\mathcal{Q}$, and maps $\mathcal{Q}_\psi$ isomorphically to $\mathcal{Q}_{f(\alpha)\psi}$. It follows that the group G_0 acts on $\mathcal{Q}_\psi$ by automorphisms, and that the induced action of $G/G_0 \cong \boldsymbol{\mu_k}$ establishes isomorphisms between $\mathcal{Q}_\psi/G_0$ and $\mathcal{Q}_{\lambda\psi}/G_0$ for $\lambda \in \boldsymbol{\mu_k}$.

The quotient space $\mathcal{Q}_\psi/G$ has only canonical singularities. By a theorem of Markushevich [20, Prop. 4] and Roan [22, Prop. 2], these singularities can be resolved to give a Calabi-Yau manifold $\mathcal{W}_\psi$. There are choices to be made in this resolution process; we do not specify a choice. By another theorem of Roan [23, Lemma 4], any two resolutions differ by a sequence of flops.

k	$(k_0, \ldots, k_4)$	$Q(x, \psi)$
5	$(1,1,1,1,1)$	$x_0^5 + x_1^5 + x_2^5 + x_3^5 + x_4^5 - 5\psi x_0 x_1 x_2 x_3 x_4$
6	$(2,1,1,1,1)$	$x_0^3 + x_1^6 + x_2^6 + x_3^6 + x_4^6 - 6\psi x_0 x_1 x_2 x_3 x_4$
8	$(4,1,1,1,1)$	$x_0^2 + x_1^8 + x_2^8 + x_3^8 + x_4^8 - 8\psi x_0 x_1 x_2 x_3 x_4$
10	$(5,2,1,1,1)$	$x_0^2 + x_1^5 + x_2^{10} + x_3^{10} + x_4^{10} - 10\psi x_0 x_1 x_2 x_3 x_4$

TABLE 1. The hypersurfaces.

Note that the differential form Ω from the previous section transforms as $\Omega \mapsto (\prod \alpha_j)\Omega$ under the action of $\alpha \in G$. Thus, the rational differential

$$\omega_1 = \frac{\psi\Omega}{Q(x, \psi)}$$

is invariant under the action of G; we define $\omega(\psi) = \mathrm{Res}_{\mathcal{Q}_\psi}(\omega_1)$.

Since the holomorphic 3-forms $\omega(\psi)$ on $\mathcal{Q}_\psi$ are invariant on G_0, they induce holomorphic 3-forms on $\mathcal{W}_\psi$. Moreover, the homology group $H_3(\mathcal{W}_\psi)$ contains the G_0-invariant part $H_3(\mathcal{Q}_\psi)^{G_0}$ of the homology of $\mathcal{Q}_\psi$. If we know that the dimensions of these spaces agree, then they will coincide (at least for homology with coefficients in a field). In this case, the periods of $\mathcal{W}_\psi$ can actually be computed as periods of the holomorphic form $\omega(\psi)$ on $\mathcal{Q}_\psi$, over G_0-invariant cycles. Thanks to the isomorphisms between $\mathcal{Q}_\psi$ and $\mathcal{Q}_{\lambda\psi}$ for $\lambda \in \boldsymbol{\mu_k}$ and the invariance of the rational differential ω_1 under G, these periods will be invariant under $\psi \mapsto \lambda\psi$. In particular, they will be functions of $z = \psi^{-k}$ alone.

It is likely that the resolutions $\mathcal{W}_\psi$ of $\mathcal{Q}_\psi/G_0$ could be chosen so that the action of G/G_0 would lift to isomorphisms between $\mathcal{W}_\psi$ and $\mathcal{W}_{\lambda\psi}$. (We verified this in the case of quintic hypersurfaces in [21].) In this case, there would be an actual family of Calabi-Yau threefolds for which z served as a parameter. It may be that such resolutions could be constructed by finding an appropriate partial resolution of $\mathcal{Q}/G$. However, we do not need the existence of this family to describe the computation of the Yukawa coupling.

We will carry out the computation in four specific examples. These come from the lists of Candelas, Lynker and Schimmrigk [6]; they found that there are exactly four types of hypersurface in weighted projective four-space which are Calabi-Yau threefolds with Picard number one. The weights of the space are given in the second column of table 1. For each of those cases, Greene and Plesser's mirror construction [13] yields the family $\mathcal{W}_\psi$ which we have described above. And Roan's formula [24] for the Betti numbers verifies that b_3 is indeed 4 (with $h^{2,1} = 1$).

The remaining columns in table 1 show the value of k, and give the equation $Q(x, \psi)$ explicitly.

We describe the G_0-invariant cohomology by means of the rational differential forms

$$\omega_\ell := \frac{(-1)^{\ell-1}(\ell-1)!\, \psi^\ell (\prod x_i^{\ell-1})\Omega}{Q(x,\psi)^\ell}.$$

These are chosen because of the evident G-invariance in the numerator; the coefficients were adjusted so that the formula

$$-\frac{1}{k}\psi\frac{d}{d\psi}\omega_\ell = -\frac{\ell}{k}\omega_\ell + \omega_{\ell+1} \tag{7}$$

would not be overly burdened with constants. We compute with the differential operator $-\frac{1}{k}\psi\frac{d}{d\psi}$ because it coincides with $z\frac{d}{dz}$.

A basis for the G_0-invariant cohomology is then given by the residues of ω_1, ω_2, ω_3, ω_4. To compute the Picard-Fuchs equation, we must find an expression for ω_5 as a linear combination of $\omega_1, \ldots, \omega_4$ modulo exact forms. That expression, combined with (7), will then yield the desired differential equation.

We carried out this calculation using the Gröbner basis algorithm [3], modifying an implementation written in MAPLE by Yunliang Yu (cf. [28]). We first calculated a Gröbner basis for the Jacobian ideal $J = (\partial Q/\partial x_0, \ldots, \partial Q/\partial x_4)$, working in the ring $\mathbb{C}(\psi)[x_0, \ldots, x_4]$ of polynomials whose coefficients are rational functions of ψ. The reduction of pole order was then achieved step by step as follows: given a form η_ℓ, the residue of a global form with a pole of order ℓ, we used the Gröbner basis to reduce the numerators of both η_ℓ and ω_ℓ to standard form. We could thus determine a coefficient $\varepsilon_\ell \in \mathbb{C}(\psi)$ such that the numerator of $\eta_\ell - \varepsilon_\ell\omega_\ell$ lies in J. Another application of Gröbner basis reduction produced explicit coefficients

$$\eta_\ell - \varepsilon_\ell\omega_\ell = \sum A_{\ell j}\frac{\partial Q}{\partial x_j}.$$

Then the Griffiths formula (6) determines forms φ_ℓ and $\eta_{\ell-1}$ such that

$$\eta_\ell - \varepsilon_\ell\omega_\ell = d\varphi_\ell + \eta_{\ell-1},$$

and $\eta_{\ell-1}$ has a pole of order $\ell - 1$.

Beginning with $\eta_5 = \omega_5$ and applying this procedure several times, one finds

$$\omega_5 = \varepsilon_1\omega_1 + \cdots + \varepsilon_4\omega_4 + d\varphi.$$

The results of this computation for our four examples are summarized in table 2. The coefficients ε_ℓ are in fact functions of $z = \psi^{-k}$ (as expected from our earlier discussion), and have been displayed as such.

k	ε_1	ε_2	ε_3	ε_4
5	$\dfrac{1}{625(z-1)}$	$\dfrac{-3}{25(z-1)}$	$\dfrac{1}{(z-1)}$	$\dfrac{-2}{(z-1)}$
6	$\dfrac{1}{324(z-4)}$	$\dfrac{-5}{18(z-4)}$	$\dfrac{-(z-50)}{18(z-4)}$	$\dfrac{-(z+20)}{3(z-4)}$
8	$\dfrac{1}{16(z-256)}$	$\dfrac{-15(z+256)}{512(z-256)}$	$\dfrac{-5(3z-1280)}{64(z-256)}$	$\dfrac{-(3z+1280)}{4(z-256)}$
10	$\dfrac{5}{4(z-12500)}$	$\dfrac{-(7z+37500)}{200(z-12500)}$	$\dfrac{-(7z-62500)}{20(z-12500)}$	$\dfrac{-(z+12500)}{(z-12500)}$

TABLE 2. The results of the Gröbner basis calculation.

The differential equation for $[\omega_1, \ldots, \omega_4]$ determined by this procedure has the form

$$z\frac{d}{dz}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix} = \begin{bmatrix}-\frac{1}{k} & 1 & 0 & 0\\ 0 & -\frac{2}{k} & 1 & 0\\ 0 & 0 & -\frac{3}{k} & 1\\ \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4-\frac{4}{k}\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}.$$

To calculate the Picard-Fuchs equation, we must change basis via

$$\begin{bmatrix}\omega_1\\ z\frac{d}{dz}\omega_1\\ (z\frac{d}{dz})^2\omega_1\\ (z\frac{d}{dz})^3\omega_1\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & 0\\ -\frac{1}{k} & 1 & 0 & 0\\ \frac{1}{k^2} & -\frac{3}{k} & 1 & 0\\ -\frac{1}{k^3} & \frac{7}{k^2} & -\frac{6}{k} & 1\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\\ \omega_4\end{bmatrix}.$$

This determines an equation in the form (4), with

$$\begin{aligned}
B_0(z) &= -\varepsilon_1(z) - \tfrac{1}{k}\varepsilon_2(z) - \tfrac{2}{k^2}\varepsilon_3(z) - \tfrac{6}{k^3}\varepsilon_4(z) + \tfrac{24}{k^4}\\
B_1(z) &= -\varepsilon_2(z) - \tfrac{3}{k}\varepsilon_3(z) - \tfrac{11}{k^2}\varepsilon_4(z) + \tfrac{50}{k^3}\\
B_2(z) &= -\varepsilon_3(z) - \tfrac{6}{k}\varepsilon_4(z) + \tfrac{35}{k^2}\\
B_3(z) &= -\varepsilon_4(z) + \tfrac{10}{k}.
\end{aligned} \tag{8}$$

As can be directly verified in each of our cases, $B_j(0) = 0$. It follows that the monodromy at $z = 0$ is maximally unipotent. (In the case of quintics ($k = 5$), this had been shown in [5]; cf. [21].)

5. Examples: Mirror maps

We next compute the mirror maps for our four examples, based on their Picard-Fuchs equations. Expanding eqs. (2) and (3), one finds that the coefficient $C_3(z)$ coincides with $(6 + B_3(z))/z$. Moreover, in our four examples, a straightforward computation based on eq. (8) and table 2 shows that $B_3(z) = 2z/(z - \lambda)$, where $\lambda = 1$, 4, 256, 12500 when $k = 5$, 6, 8, 10, respectively. Thus,

$$C_3(z) = \frac{6 + B_3(z)}{z} = \frac{6}{z} + \frac{2}{z - \lambda}.$$

The Yukawa coupling κ_{zzz} in the gauge $\omega(z)$ is therefore given by a function $W_3(z)$ which satisfies the differential equation

$$\frac{dW_3(z)}{dz} = \left(\frac{-3}{z} + \frac{-1}{z - \lambda}\right) W_3(z).$$

Thus, in the gauge $\omega(z)$ we have

$$\kappa_{zzz} = \frac{c_1}{(2\pi i)^3 z^3 (z - \lambda)}.$$

Here $c_1/(2\pi i)^3$ is the first "constant of integration": we have introduced a factor of $(2\pi i)^3$ in order to simplify a later formula.

In order to determine the natural gauge, we must find a solution $f_0(z)$ of the Picard-Fuchs equation which is regular near $z = 0$. Using the corresponding vector $w_0(z)$ of which $f_0(z)$ is the first component, we want a solution to the vector equation

$$z\frac{d}{dz} w_0(z) = A(z) w_0(z) \tag{9}$$

which is regular near $z = 0$. ($A(z)$ is given by eqs. (4), (8), and table 2.) This can be found using power-series techniques, and there is a solution with $f_0(0) \neq 0$ in each of our four cases. We normalize so that $f_0(0) = 1$; alternatively, we could have absorbed the leading term of $f_0(z)$ into the constant of integration c_1.

As a result, the gauge-fixed value of κ_{zzz} takes the form

$$\kappa_{zzz} = \frac{c_1}{(2\pi i)^3 z^3 (z - \lambda)(f_0(z))^2},$$

where the constant c_1 has yet to be determined.

We now search for the good parameter t. We should locate a second solution $f_1(z)$, or its corresponding vector $w_1(z)$, which is multiple-valued and has the correct monodromy properties. The monodromy will be such that if we introduce

$$v(z) := 2\pi i w_1(z) - (\log z) w_0(z)$$

and its first component

$$g(z) := 2\pi i f_1(z) - (\log z) f_0(z),$$

then $v(z)$ will be single-valued and regular near $z = 0$. It is easy to calculate that the matrix equation satisfied by $v(z)$ is

$$z\frac{d}{dz}v(z) = A(z)v(z) - w_0(z). \tag{10}$$

Solutions to this equation can be found by power-series techniques. We normalize the solution so that $g(0) = 0$. The parameter t is then given by

$$t = \frac{1}{2\pi i}\log c_2 + \frac{1}{2\pi i}\log z + \frac{g(z)}{f_0(z)}$$

($\frac{1}{2\pi i}\log c_2$ is the second "constant of integration") and the associated parameter q is

$$q = e^{2\pi i t} = c_2 z e^{g/f_0}.$$

Let us define

$$\delta(z) = 1 + z\frac{d}{dz}\left(\frac{g(z)}{f_0(z)}\right),$$

so that

$$\frac{dq}{dz} = c_2\delta(z)e^{g/f_0}.$$

Then by the chain rule,

$$\frac{dz}{dt} = \frac{dq/dt}{dq/dz} = \frac{2\pi i z}{\delta(z)}.$$

It follows that the gauge-fixed value of κ_{ttt} is

$$\kappa_{ttt} = \left(\frac{dz}{dt}\right)^3 \kappa_{zzz} = \frac{c_1}{(\delta(z))^3(z-\lambda)(f_0(z))^2}.$$

Finally we express this normalized κ_{ttt} as a power series in q. The constants c_1 and c_2 have yet to be determined; however, we can define

$$h_0(z) = \frac{1}{(\delta(z))^3(z-\lambda)(f_0(z))^2} \tag{11}$$

$$h_j(z) = \frac{1}{\delta(z)e^{g/f_0}} \cdot \frac{dh_{j-1}(z)}{dz} \tag{12}$$

and find that

$$h_j(z) = \frac{(c_2)^j}{c_1}\left(\frac{d}{dq}\right)^j \kappa_{ttt},$$

so that

$$\kappa_{ttt} = \sum_{j=0}^{\infty} \frac{c_1}{(c_2)^j}\,\frac{h_j(0)}{j!}\,q^j.$$

Proposition . *The numbers $h_j(0)$ are rational numbers.*

Proof. The coefficient matrix $A(z)$ in the vector equation (9) has entries in $\mathbb{Q}(F)$; if written out in power series, all the power series coefficients will be rational numbers. Finding a power series solution to (9) then involves solving linear equations with rational coefficients at each step: the solutions will be rational. Thus, $w_0(z)$ and $f_0(z)$ are power series in z with rational coefficients.

Similarly, $v(z)$ and $g(z)$ are power series with rational coefficients, since they come from equation (10). Furthermore, since exponentiating a power series with rational coefficients (whose constant term is zero) again gives a power series with rational coefficients, e^{g/f_0} and $\delta(z)$ are power series in z with rational coefficients.

But then by (11), $h_0(z)$ is clearly a power series in z with rational coefficients; similarly for $h_j(z)$ by (12). It follows that each $h_j(0)$ is a rational number. Q.E.D.

6. Choosing the constants and predicting the numbers of rational curves

Calabi-Yau threefolds with $h^{2,1} = 1$ are conjectured to be the "mirrors" of other Calabi-Yau threefolds with $h^{1,1} = 1$. In the four examples we have considered, this mirror property can be realized by a construction of Greene and Plesser [13]. The threefolds $\mathcal{W}_\psi$ are mirrors of threefolds $\mathcal{M} \subset \mathbb{P}^{(k_1,\dots,k_5)}$, which are hypersurfaces of weighted degree $k = \sum k_j$. The Picard group of $\mathcal{M}$ is cyclic, generated by some ample divisor H.

Mirror symmetry predicts that the q-expansion of the gauge-fixed Yukawa coupling

$$\kappa_{ttt} = a_0 + a_1 q + a_2 q^2 + \cdots$$

will have integers as coefficients. Moreover, by a formula conjectured in [5] and established in [1], if this q-expansion is written in the form

$$\kappa_{ttt} = n_0 + \sum_{j=1}^{\infty} \frac{n_j j^3 q^j}{1-q^j} = n_0 + n_1 q + (2^3 n_2 + n_1)q^2 + \cdots . \tag{13}$$

k	n_0	n_1	n_2	n_3	n_4
5	5	2875	609250	317206375	242467530000
6	3	7884	6028452	11900417220	34600752005688
8	2	29504	128834912	1423720546880	23193056024793312
10	2	462400	24431571200	3401788732948800	700309317702649312000

TABLE 3. The predicted numbers of curves.

then the coefficients n_j are also integers. The first term n_0 is predicted to coincide with H^3 (the absolute degree of $\mathcal{M}$), and n_j is predicted to be the number of rational curves C on $\mathcal{M}$ with $C \cdot H = j$, assuming that all rational curves on $\mathcal{M}$ are disjoint and have normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

These two predictions can be used to choose the constants of integration in our examples. First, the absolute degree d is the lowest order term which appears in the polynomial $Q(x, \psi)$; to ensure that $n_0 = d$ we must take $c_1 = -\lambda d$. Second, the formula (13) puts very strong divisibility constraints on the coefficients a_j, and it seems likely that there will be a unique choice of c_2 which satisfies all of these constraints.

We have calculated the first 20 coefficients (using MATHEMATICA) in each of our four examples. There does indeed appear to be a unique choice for c_2 which produces integers for $n_1, \ldots n_{20}$: that choice turns out to be $c_2 = k^{-k}$ in each of our examples. Making this choice leads to the values for n_j displayed in table 3.

Table 3 therefore contains predictions about numbers of rational curves on the weighted projective hypersurfaces. For a general hypersurface in $\mathcal{M} \subset \mathbb{P}^{(k_1,\ldots,k_5)}$ of degree $k = \sum k_j$, the prediction is that there should be n_j rational curves C with $C \cdot H = j$, where H generates $\mathrm{Pic}(\mathcal{M})$.

The first line of the table reproduces the predictions made by Candelas et al. about quintic threefolds. Several of these have been verified: the number of lines was known classically, the number of conics was computed by Katz [18], and the number of twisted cubics n_3 has recently been computed by Ellingsrud and Strømme [11]—all of these results agree with the predictions.

Of the remaining predictions in the table, we have only checked one. Each hypersurface from the third family (the case $k = 8$) can be regarded as a double cover of $\mathbb{P}^3$ branched on a surface of degree 8. The entry 29504 in the third line of the table can be interpreted as follows: for a general surface of degree 8 in $\mathbb{P}^3$, there should be 14752

lines which are 4-times tangent to the surface. (These lines will then split into pairs of rational curves on the double cover.) After we had obtained this number, Steve Kleiman was kind enough to locate a 19$^{\text{th}}$-century formula of Schubert [25, Formula 21, p. 236], which states that the number of lines in $\mathbb{P}^3$ 4-times tangent to a general surface of degree n is

$$\frac{1}{12}n(n-4)(n-5)(n-6)(n-7)(n^3+6n^2+7n-30).$$

Substituting $n = 8$, we find the predicted number 14752.

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Department of Mathematics, Duke University, Durham, NC 27706
E-mail address: drm@math.duke.edu